π kinks in strongly ac driven sine-Gordon systems

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(Received 16 March 1998)

We demonstrate that π kinks exist in nonparametrically ac driven sine-Gordon systems if the ac drive is sufficiently fast. It is found that, at a critical value of the drive amplitude, there are two stable and two unstable equilibria in the sine-Gordon phase. The pairwise symmetry of these equilibria implies the existence of a one-parameter family of π-kink solutions in the reduced system. In the dissipative case of the ac driven sine-Gordon systems, corresponding to Josephson junctions, the velocity is selected by the balance between the perturbations. The results are derived from a perturbation analysis and verified by direct numerical simulations. [S1063-651X(98)51607-8]

PACS number(s): 03.40.Kf, 52.35.Mw, 02.30.Jr

Soliton bearing systems are very important for our understanding of collective phenomena in many physical systems in the one-dimensional approximation [1]. Often, such systems are perturbed in one form or another and sometimes these perturbations are temporally periodic [2–4]. If the perturbations are small, one can approximate the dynamics through the adiabatic perturbation technique [5,6], where the integrability of the unperturbed system is used to assume wave profiles for analytical perturbation techniques applied to the ''nearly integrable'' system. However, if the perturbations are large, we cannot simply assume the unperturbed wave profiles as representing a good approximation to the dynamics since the near integrability is lost.

This problem was previously addressed for strong periodic perturbations of sine-Gordon systems [7,8], where it was shown that the direct ac drive can induce a Shapiro type locked phase [9,10] to which a 2π kink can localize. Similarly, it has been demonstrated [11–13] that a parametrically driven sine-Gordon system can produce stable π-kink propagation due to the well known effective Kapitza potential of a driven pendulum [14].

It is thereby well documented that strongly perturbed nonlinear soliton bearing systems can retain some forms of near integrability in certain windows of the perturbation parameter space. Previous analyses for the sine-Gordon system have been performed using the analogies to the single pendulum cases, where Shapiro phase locking exists for the direct drive, leading to stable 2π kinks in a rescaled sine-Gordon chain.

In this Rapid Communication we demonstrate that *π kinks* can propagate in strongly perturbed, directly driven sine-Gordon chains if the perturbation parameters are chosen near the values, leading to zero crossing of the relevant Bessel functions determining the size of a Shapiro step. The analysis is based on the normal form technique that relies upon a time scale separation between the rapidly oscillating driving force and a relatively slow behavior of the residual field. This technique was previously applied to the parametrically forced sine-Gordon equation (SGE) in [11,12]. We find for nondissipative ac driven SGE a one-parameter family of π kink solutions moving with any prescribed velocity. In the case of damped and driven SGE, we show that the velocity is selected and only one of the π-kinks survives. We verified the obtained results by performing numerical simulations of the ac driven SGE.

First, we consider the equation of motion for a directly forced pendulum $\ddot{\phi}+\sin\phi=Mf(\omega t)$, where ϕ is the phase of the pendulum, f is a mean-zero periodic function, M is constant, t represents a normalized time, and the normalized frequency ω is assumed to be sufficiently large compared to the natural frequency of the pendulum, $\omega_0=1$. We shift the emphasis to the oscillating reference frame by the transformation

$$\phi=\theta+M\omega^{-2}F(\omega t), \tag{1}$$

where F has zero mean and $F''(\tau)=f(\tau)$. We then obtain the parametrically forced equation $\ddot{\theta}+\sin[\theta+M\omega^{-2}F(\omega t)]=0$, with the Hamiltonian

$$H=\frac{p^2}{2}-A(\omega t)\cos(\theta)+B(\omega t)\sin(\theta), \tag{2}$$

where p is the momentum canonically conjugate to θ, and $A(\omega t)=\cos[M\omega^{-2}F(\omega t)]$, $B(\omega t)=\sin[M\omega^{-2}F(\omega t)]$. Using $2\pi/\omega$ periodicity of A,B, we denote $\{A\}=A-\langle A\rangle$, $\{B\}=B-\langle B\rangle$, where $\langle\cdots\rangle\equiv(1/2\pi)\int_0^{2\pi}d\tau\cdots$.

Following Refs. [11,12], we apply the normal form technique, to move mean-zero terms to a higher order. Let the first canonical transformation be defined implicitly as

$$p=p_1+\partial_\theta W_1(\theta,p_1,t),\quad \theta_1=\theta+\partial_{p_1}W_1(\theta,p_1,t). \tag{3}$$

The transformed Hamiltonian takes the form $H_1=H+W_{1t}$. To remove mean-zero rapidly oscillating terms, we choose $W_1=\omega^{-1}\{A\}_{-1}\cos(\theta)-\omega^{-1}\{B\}_{-1}\sin(\theta)$, where $\{A\}_{-1}$ is a mean-zero antiderivative of $\{A\}$. With this choice of W_1 the transformed Hamiltonian takes the form

$$H_1=\frac{p_1^2}{2}-\langle A\rangle\cos\theta_1+\langle B\rangle\sin\theta_1-\omega^{-1}p_1\times(\{A\}_{-1}\sin\theta_1+\{B\}_{-1}\cos\theta_1)+\frac{\omega^{-2}}{2}(\{A\}_{-1}\sin\theta_1+\{B\}_{-1}\cos\theta_1)^2. \quad (4)$$

Unlike the original Hamiltonian (2), the transformed Hamiltonian H_1 contains terms with small ($\sim\omega^{-2}$) positive time-dependent coefficients $\{A\}_{-1}^2,\{B\}_{-1}^2$, which have nonzero averages for any nontrivial choice of M,f, and therefore, cannot be removed from the Hamiltonian.

These terms have an essential effect on the system dynamics when the lower order potential energy terms vanish, i.e., $\langle A\rangle=\langle B\rangle=0$ (all terms $\sim\omega^{-1}$ always have zero averages). To remove the explicit time dependence from the Hamiltonian up to the terms $\sim\omega^{-2}$, we perform a series of canonical transformations similar to Eq. (3) (see also [12]). To remove the terms $\sim\omega^{-1}$, we apply the transformation

$$p_1=p_2+\partial_{\theta_1}W_2(\theta_1,p_2,t),\quad \theta_2=\theta_1+\partial_{p_2}W_2(\theta_1,p_2,t), \quad (5)$$

with

$$W_2=\omega^{-2}p_2[\{A\}_{-2}\sin(\theta_1)-\{B\}_{-2}\cos(\theta_1)]. \quad (6)$$

After straightforward calculations we obtain the transformed Hamiltonian

$$H_2=\frac{p_2^2}{2}-\langle A\rangle\cos\theta_2+\langle B\rangle\sin\theta_2+\frac{\omega^{-2}}{2}(\langle\{A\}_{-1}^2\rangle\sin^2\theta_2+\langle\{B\}_{-1}^2\rangle\cos^2\theta_2+\langle\{A\}_{-1}\{B\}_{-1}\rangle\sin 2\theta_2)+\omega^{-2}R+O(\omega^{-3}), \quad (7)$$

where R turns out to be mean zero $\langle R\rangle=0$. Finally, applying the third transformation with $W_3=\omega^{-3}R_{-1}$ and neglecting terms $\sim\omega^{-3}$ we obtain

$$\tilde{H}(P,\Theta)=\frac{P^2}{2}-C\cos(\Theta-\gamma)-\frac{\omega^{-2}}{2}D\cos(2\Theta-\delta), \quad (8)$$

where

$$\langle A\rangle=C\cos(\gamma),\quad \langle\{A\}_{-1}^2\rangle-\langle\{B\}_{-1}^2\rangle=2D\cos(\delta),$$
$$\langle B\rangle=C\sin(\gamma),\quad -\langle\{A\}_{-1}\{B\}_{-1}\rangle=D\sin(\delta), \quad (9)$$

and P and Θ are new canonical variables.

For $C\neq 0$ there is only one stable equilibrium $\Theta=\gamma$ and one unstable equilibrium $\Theta=\pi+\gamma$, for large frequencies; this equilibrium corresponds to the usual phase-locked Shapiro state known from Josephson junctions [9]. However, as C passes through 0, a bifurcation occurs and (for $C=0$) the system has two stable equilibria given by $\Theta=\delta/2$, $\pi+\delta/2$ and two unstable equilibria given by $\Theta=\pi/2+\delta/2$, $3\pi/2+\delta/2$.

Now we turn to the *directly forced SGE*

$$\phi_{tt}-\phi_{xx}+\sin\phi=Mf(\omega t). \quad (10)$$

After applying the transformation (1) we obtain the evolution equation for a new phase θ on top of a rapidly oscillating background field (written in the canonical form)

$$\theta_t=p,\quad p_t=\theta_{xx}-\sin[\theta+M\omega^{-2}F(\omega t)]. \quad (11)$$

Invoking the canonical transformations similar to those for the directly forced pendulum (see also [12]) and using Eq. (8), we obtain for the corresponding Hamiltonian

$$H=\int_{-\infty}^{+\infty}\left[\frac{\Theta_x^2}{2}+\tilde{H}(P,\Theta)+O(\omega^{-3})\right]dx, \quad (12)$$

where the error terms $O(\omega^{-3})$ contain the derivatives up to the second order. For sufficiently large ω, when we can neglect these terms, the above Hamiltonian corresponds to the double SGE. After retracing the identical transformation (1), the obtained approximate solutions become π kinks on top of the rapidly oscillating background field.

We now consider the *damped and driven SGE*

$$\phi_{tt}-\phi_{xx}+\sin\phi=Mf(\omega t)-\alpha\phi_t+\eta, \quad (13)$$

which is frequently used to describe long Josephson junctions [6], where ϕ is the phase difference between the quantum mechanical wave functions of the two superconductors defining the junction, the normalized time t is measured relative to the inverse plasma frequency, space x is normalized to the Josephson penetration depth, and the nonlinear term represents tunneling of superconducting Cooper pairs, normalized to the critical current density. The perturbations on the right-hand side of the equation represent, respectively, a normalized ac driving current, a dissipative term arising from tunneling of quasiparticles, and a normalized dc bias current.

To obtain an effective equation of the evolution on the slow time scale, we apply the above canonical transformations. Since the system is no longer Hamiltonian, we work with equations of motion rather than Hamiltonians. We start with a homogeneous transformation to the oscillating reference frame, $\phi=\theta+G(t)$, analogous to Eq. (1), designed to remove the free oscillatory term. Substituting this transformation to Eq. (13) and choosing the function G so that it solves the equation $\ddot{G}+\alpha\dot{G}=Mf(\omega t)$, we obtain the equations of motion in the canonical form

$$\theta_t=p,\quad p_t=\theta_{xx}-\alpha p+\eta-\sin[\theta+G(\omega t)]. \quad (14)$$

For the particular case of $f(\tau)=\sin\tau$, we find

$$G(\tau)=-\frac{\alpha}{\omega}\frac{M}{\alpha^2+\omega^2}\cos\tau-\frac{M}{\alpha^2+\omega^2}\sin\tau. \quad (15)$$

Using the notations $A=\cos G(\omega t)$, $B=\sin G(\omega t)$ and assuming that $\langle A\rangle=\langle B\rangle=0$, we apply the series of transformations, as we did for the directly forced pendulum. This moves all mean-zero terms to higher order, leading to

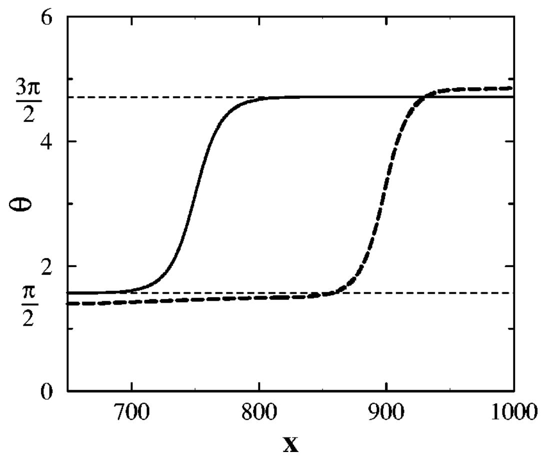

FIG. 1. Behavior of π-kink solution in the ac driven SGE (11). The solid line corresponds to the initial profile given by Eq. (19), with $c=0.5$ and $F(\omega t)=\sin(\omega t)$. The dashed line corresponds to $t=300$. The parameters are $\omega=10$, time step $dt=0.01$, mesh size $dx=0.1$, system size $L=1500$.

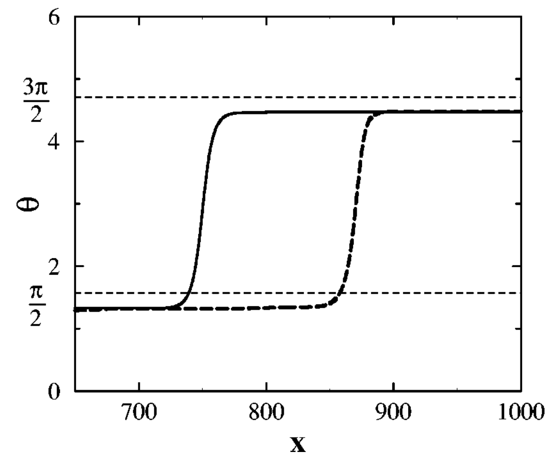

FIG. 2. Behavior of π-kink solution in the damped and driven SGE (14). The solid line corresponds to the initial profile $\theta(x,0)=V(x,0)+\arcsin(\omega^2\eta/D)$, with V given by Eq. (19) and $G(\omega t)$ given by Eq. (15). The dashed line corresponds to $t=150$. The parameters are $\omega=6$, time step $dt=0.01$, mesh size $dx=0.1$, system size $L=1500$, $\alpha=0.03$, $\eta=-0.003$.

$$\Theta_t=P+O(\omega^{-3}),$$

$$P_t=\Theta_{xx}-\alpha P+\eta-\omega^{-2}D\ \sin(2\Theta-\delta)+O(\omega^{-3}), \tag{16}$$

where D and δ are given by Eq. (9). Constants α and η in Eq. (16) are assumed to be sufficiently small, so that the corresponding terms could be considered as a perturbation. Then, in zeroth order in α,η, the system (16) reduces (after neglecting terms $\sim\omega^{-3}$) to SGE, which has π-kink solutions. Therefore, slightly perturbed π kinks are approximate solutions of the original equation (13) on top of the rapidly oscillating background field.

To verify the above predictions, we have performed a set of numerical simulations. Rescaling the variables in equations of motion (16) [which in the Hamiltonian case correspond to the Hamiltonian (12)], by

$$\chi=2\Theta-\delta,\quad p_\chi\frac{\sqrt{D/2}}{\omega}=P,$$

$$X=\frac{\sqrt{2D}}{\omega}x,\quad T=\frac{\sqrt{2D}}{\omega}t, \tag{17}$$

we obtain the sine-Gordon system

$$\chi_T=p_\chi,\quad p_{\chi T}=\chi_{XX}-\alpha\frac{\omega}{\sqrt{2D}}p_\chi+\frac{\omega^2}{D}\eta-\sin\chi. \tag{18}$$

In the *Hamiltonian case* ($\alpha=\eta=0$), Eq. (18) has solitary wave solutions which, after retracing the transformation (17), read $\Theta=V(x,t)$, $P=V_t(x,t)$, where

$$V(x,t)=\frac{\delta}{2}+2\arctan\left[\exp\frac{\sqrt{2D}}{\omega}\left(\frac{x-ct}{\sqrt{1-c^2}}\right)\right]. \tag{19}$$

Using Eq. (3) we return to the original variables, which gives us the approximate solution

$$\theta=V,\quad p=V_t-\frac{\{A\}_{-1}}{\omega}\sin(V)-\frac{\{B\}_{-1}}{\omega}\cos(V), \tag{20}$$

which we use to generate initial conditions for Eq. (11).

We have performed numerical simulations of Eq. (11), which is equivalent to the original directly forced SGE (10). We used a second-order leap-frog method, with initial conditions given by Eq. (20). The coefficients D and δ in Eq. (19) are calculated from Eq. (9): $D=0.2270596\cdots$; $\delta\approx\pi$. Figure 1 shows the results of the simulations. The driving amplitude M was chosen so as to make both $\langle A\rangle$ and $\langle B\rangle$ vanish: $M\omega^{-2}\approx 2.4048\cdots$. The velocity c in Eq. (19) was taken as $c=0.5$. The small parameter used in our perturbation analysis is $\epsilon=\omega^{-1}$. One can see from the figure that the kink indeed moves with the velocity $c\approx 0.5$ and that it is stable for $t\leqslant 300$, much longer than ϵ^{-1} (in the considered case $\epsilon=0.1$), which is a natural estimate for the validity of averaging for time-periodic perturbation based on multiple scale procedure. For longer times ($t\geqslant 300$), the π kink becomes unstable and eventually disintegrates due to the parametric resonances. A similar destruction of the π kink after a long time was observed in a parametrically excited SGE [11,12].

In the *dissipative case* ($\alpha\neq 0$ and $\eta\neq 0$), we have shown that Eq. (13) reduces to Eq. (16), after averaging over the fast time scale and neglecting terms of order ω^{-3}. Then only one π-kink solution of Eq. (16), with a certain value of c, is selected out of the entire family, because of the energy balance consideration.

To find the selected c, we substitute a traveling wave ansatz, $Z=X-cT$, into Eq. (18). In the zeroth order in α and η, we obtain $\chi_0=4\arctan[\exp(Z/\sqrt{1-c^2})]$. The solvability condition for the linearized equation for the first order correction χ_1, gives us the velocity

$$c=-\frac{\sqrt{2}\omega\eta}{\alpha\sqrt{D}}\frac{\int_{-\infty}^{+\infty}\chi_0' dZ}{\int_{-\infty}^{+\infty}(\chi_0')^2 dZ}=-\frac{\pi\eta\omega}{\sqrt{8\alpha^2D+\pi^2\eta^2\omega^2}}. \tag{21}$$

We have simulated Eq. (14), equivalent to the original forced and damped SGE (13), with the initial conditions obtained from Eq. (20) by adding a small correction $\arcsin(\omega^2\eta/D)$ to the first equation of Eq. (20), to compensate an additional energy transferred to the kink from the constant source in Eq. (14). The results of the simulations are given in Fig. 2. The π kink in the figure moves with the velocity $c \approx 0.79$, which approximately coincides with the selected velocity $c \approx 0.81$ obtained from Eq. (21). The plotted solution remained intact for times $t \leqslant 150$. Shortly after that it is destroyed by higher order effects such as radiation and nonlinear resonances.

We have demonstrated that directly strongly ac driven SGE can produce localized π kinks in special regions of parameter space. These regions coincide with the regions where 2π-kink localization (Shapiro steps) vanishes. The formalism for demonstrating the existence of the π kinks is based on a time scale separation technique, where the ac drive is assumed to be fast compared to any natural oscillation in the unperturbed SGE. As a consequence, the predicted localization mechanism is limited to the high frequency region of parameter space. As the driving frequency is lowered, nonlinear mixing between the supposed high frequency drive and low frequency wave dynamics become more dominant, eventually leading to the destruction of localization and all coherent behavior. We have numerically tested that long time propagation of π kinks is possible for relatively low driving frequencies, as small as $\omega \sim 5$, and moderate driving amplitudes, $M \sim 60$. While technical Josephson applications of π-kink localization seem far removed from the present, the above perturbation parameters suggest that the predicted effect may exist within experimental parameters used for generating Shapiro steps in current-voltage characteristics of Josephson junctions.

This work was performed under the auspices of the U.S. Department of Energy. The work of V.Z. was supported by the NSF under Grant No. DMS-9627721.

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